Linear Algebra Done Right

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1 Vector Spaces

1.1 R^n and C^n

1.1.1 Complex Numbers

A ***complex number*** is an ordered pair (a, b), where $a, b \in R$, but it is denoted as $a + bi$.

The set of all complex numbers is denoted by C:

$$C = \{a + bi : a, b \in R\}.$$

Addition and multiplication on C are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i,$$

where $a, b, c, d \in R$.

Intuitively, $a + 0i$ is the real number a. Hence, R is a subset of C.

1.1.2 Properties of complex arithmetic

- **commutativity**
  $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in C$.
- **associativity**
  $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all $\alpha, \beta, \gamma \in C$.
- **identities**
  $\gamma + 0 = \gamma$ and $\gamma \cdot 1 = \gamma$ for all $\gamma \in C$.
- **additive inverse**
  for every $\alpha \in C$, there exists a unique $\beta \in C$ such that $\alpha + \beta = 0$.
- **multiplicative inverse**
  for every $\alpha \in C$, there exists a unique $\beta \in C$ such that $\alpha\beta = 1$.
- **distributive property**
  $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$ for all $\gamma, \alpha, \beta \in C$.

Notation: Throughout these notes, F stands for either R or C. This is used because R and C are examples of what are called **fields**.

Elements of F are called **scalars**, emphasizing that an object is a number as opposed to a vector.

For $\alpha \in F$ and m positive integer, we define α^m to denote the product of α with itself m times:

$$\alpha^m = \underbrace{\alpha \cdots \alpha}_{m \text{ times}}.$$

Clearly $(\alpha^m)^n = \alpha^{mn}$ and $(\alpha\beta)^m = \alpha^m \beta^m$ for all $\alpha, \beta \in F$ and all positive integers m, n.

1.1.3 Lists

Before defining R^n and C^n , we look at two important examples:

- The set R^2, which you can think of as a plane, is the set of all ordered pairs of real numbers:

$$R^2 = \{(x, y) : x, y \in R\}.$$

- The set R^3, which you can think of as ordinary space, is the set of all ordered triples of real numbers:

$$R^3 = \{(x, y, z) : x, y, z \in R\}.$$

For a nonnegative integer n, a ***list*** of ***length*** n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this:

$$(x_1, \ldots, x_n).$$

Two lists are equal if and only if they have the same length and the same elements *in the same order*.

Many mathematicians call a list of length n an n-tuple. Also remember that a list has a finite length, thus $(x_1, x_2, \ldots)$, is not a list.

A list of length 0 looks like this: (). We consider such an object to be a list to avoid trivial exceptions.

Lists differ from sets in two ways: in lists, order matters and repetitions have meaning; in sets, order and repetitions are irrelevant. For example,

- the lists $(3, 5)$ and $(5, 3)$ are not equal, but the sets $\{3, 5\}$ and $\{5, 3\}$ are equal.

- The lists $(4, 4)$ and $(4, 4, 4)$ are not equal (they do not have the same length), although the sets $\{4, 4\}$ and $\{4, 4, 4\}$ both equal the set $\{4\}$.

1.1.4 F^n

F^n is the set of all lists of length n of elements of F:

$$F^n = \{(x_1, \ldots, x_n) : x_j \in F \text{ for } j = 1, \ldots, n\}.$$

For $(x_1, \ldots, x_n) \in F^n$ and $j \in \{1, \ldots, n\}$, we say that x_j is the j^{th} ***coordinate*** of $(x_1, \ldots, x_n)$. For example, C^4 is the set of all lists of four complex numbers:

$$C^4 = \{(z_1, z_2, z_3, z_4) : z_1, z_2, z_3, z_4 \in C\}.$$

Visualizing high dimensional sets is difficult, but we can perform algebraic manipulations in F^n as easily as in R^2 or R^3. For example, ***addition*** in F^n is defined by adding corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n).$$

Commutativity of addition in F^n If $x, y \in F^n$, then $x + y = y + x$.
The proof is trivial af bruv.

Notation: For $x \in F^n$, letting $x = (x_1, \ldots, x_n)$ is a good notation. Better to not get into coordinates and work with just x.

Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \ldots, 0)\,.$$

1.1.5 *Additive inverse in* F^n

For $x \in F^n$, the ***additive inverse*** of x, denoted $-x$, is the vector $-x \in F^n$ such that

$$x + (-x) = 0.$$

In other words, if $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$.

Visually, for a vector $x \in R^2$, the additive inverse $-x$ is the vector parallel to x and with the same length as x but pointing in the opposite direction.

1.1.6 *Scalar multiplication in* F^n

The product of a number λ and a vector in F^n is computed by multiplying each coordinate of the vector by λ:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n);$$

here $\lambda \in F$ and $(x_1, \ldots, x_n) \in F^n$.

The vector λx where λ is an integer, and x is a vector in R^2 has the magnitude $\mid \lambda \mid \cdot \mid x \mid$ and direction as of x.

1.1.7 Digression on Fields

A field is a set containing at least two distinct elements 0 and 1, along with operations of addition and multiplication satisfying all the properties listed in 1.1.3.

Examples include R, C, and the set of rational numbers along with the usual operations of addition and multiplication. Interestingly, a set $\{0, 1\}$ is also a field with usual operations of addition and multiplication except that $1 + 1$ is defined to equal 0.

1.2 Exercises

1. Suppose a and b are real numbers, not both 0. Find real numbers c and d such that
$$1/(a + bi) = c + di.$$

2. Show that
$$\frac{-1 + \sqrt{4}i}{2}$$
is a cube root of 1.

   Soln:
$$\left(\frac{-1 + \sqrt{3}i}{2}\right)^2 = \frac{-1 - \sqrt{3}i}{2}$$
$$\implies \frac{-1 - \sqrt{3}i}{2} \cdot \frac{-1 + \sqrt{3}}{2} = 1.$$

3. Find two distinct square roots of i.

   For this, we use the fact that $i = e^{i\pi/2}$.

2 Vector Space

An addition on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.

A scalar multiplication on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in F$ and each $v \in V$.

Vector Space A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

- **commutativity**
  $u + v = v + u$ for all $u, v \in V$.

- **associativity**
  $(u+v)+w = u+(v+w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in F$.

- **additive identity**
  there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in F$

- **additive inverse**
  for every $v \in V$, there exists $w \in V$ such that $v + w = 0$.

- **multiplicative identity**
  $1v = v$ for all $v \in V$.

- **distributive properties**
  $a(u+v) = au+av$ and $(a+b)v = av+bv$ for all $a, b \in F$ and all $u, v \in V$.

Elements of a vector space are called vectors or points.

In order to be precise, we say that V is a vector space over F instead of saying simply that V is a vector space. For example, R^n is a vector space over R, and C^n is a vector space over C.

Real vector space, complex vector space A vector space over R is called a real vector space, and a vector space over C is called a complex vector space.

The simplest vector space contains only one point. In other words, $\{0\}$ *is a vector space.*

2.1 F^S

- If S is a set, then F^S denotes the set of functions from S to F.
- For $f, g \in F^S$, the ***sum*** $f + g \in F^S$ is the function defined by

  $$(f+g)(x) = f(x) + g(x)$$

  for all $x \in S$.
- For $\lambda \in F$ and $f \in F^S$, the ***product*** $\lambda f \in F^S$ is the function defined by

  $$(\lambda f)(x) = \lambda f(x)$$

  for all $x \in S$.

For example, if S is the interval $[0,1]$ and $F = R$ then $R^{[0,1]}$ is the set of real-valued functions on the interval $[0,1]$.

Example: F^S ***is a vector space***

- If S is a nonempty set, then F^S (with the operations of addition and scalar multiplication as defined above) is a vector space over F.
- The additive identity of F^S is the function $0 : S \to F$ defined by

$$0(x) = 0$$

  for all $x \in S$.
- For $f \in F^S$, the additive inverse of f is the function $-f : S \to F$ defined by

$$(-f)(x) = -f(x)$$

  for all $x \in S$.
- Basically, same properties hold but with a domain S and for the set of functions F^S.

The elements of a vector space $R^{[0.1]}$ are ***real-valued functions on*** $[0, 1]$, not lists. In general, a vector space is an abstract entity whose elements might be lists, functions, or weird objects.

Note that F^n and F^∞ are special cases of vector space F^S because a list of length n of numbers in F can be thought of as a function from $\{1, 2, \ldots, n\}$ to F and a sequence of numbers in F can be thought of as a function from the set of positive integers to F.

In other words, we can think of F^n as $F^{\{1,2,\ldots,n\}}$ and F^∞ as $F^{\{1,2,\ldots\}}$.

2.1.1 Unique additive identity

A vector space has a single unique additive identity.

Proof Suppose 0 and $0'$ are both additive identities for some vector space V. Then

$$0' = 0' + 0 = 0 + 0' = 0,$$

where the first equality holds because 0 is an additive identity, the second equality comes from the commutativity, and the third holds because $0'$ is an additive identity. Thus $0' = 0$, proving that V has only one additive identity.

2.1.2 Unique additive inverse

Every element in a vector space has a unique additive inverse.

Proof Suppose V is a vector space. Let $v \in V$. Suppose w and w' are additive inverses of v. Then

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'.$$

Thus $w = w'$, as desired.

Using 2.1.1 and 2.1.2, the following notation is:

Notation $-v, w - v$ Let $v, w \in V$. Then

- $-v$ denotes the additive inverse of v;
- $w - v$ is defined to be $w + (-v)$.

For the rest of the notes, V will denote a vector space over F.

The number 0 **times a vector** $0v = 0$ for every $v \in V$, where 0 is a scalar on the LHS and a vector on the RHS.

Proof For $v \in V$, we have

$$0v = (0 + 0)v = 0v + 0v.$$

Adding the additive inverse of $0v$ to both sides of the equation above gives $0 = 0v$, as desired.

A number times the vector 0 $a0 = 0$ for every $a \in F$.

Proof For $a \in F$, we have

$$a0 = a(0 + 0) = a0 + a0.$$

Adding the additive inverse of $a0$ to both sides of the equation gives $0 = a0$, as desired.

The number -1 **times a vector** $(-1)v = -v$ for every $v \in V$.

Proof For $v \in V$, we have

$$v + (-1)v = 1v + (-1)v = (1 + (-1))\, v = 0v = 0.$$

This equation says that $(-1)v$, when added to v, gives 0. Thus $(-1)v$ is the additive inverse of v, as desired.

2.2 Exercises

1. Suppose $a \in F, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

   Soln: If $a = 0$, we are done. If $a \neq 0$, then a has inverse a^{-1} s.t. $a(a^{-1}) = 1$. So,

   $$v = 1 \cdot v = (aa^{-1})v = a^{-1}(av) = 0.$$

2. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Soln: Let $\exists x, x' \in V, v + 3x = v + 3x' = w$. Thus, $3(x - x') = w - v \implies 3x - 3x' = w - v - (w - v) = 0$. Hence, $x - x' = 0$, that is $x = x'$. This shows uniqueness.

3 Subspaces

A subset U of V is called a ***subspace*** of V if U is also a vector space (using the same addition and scalar multiplication as on V). For example, $\{(x_1, x_2, 0 : x_1, x_2 \in F\}$ is a subspace of F^3.

To check whether a subset of a vector space is a subspace, we condition the subset to the following conditions.

3.0.1 Conditions for a subspace

A subset U and V is a subspace of V if and only if U satisfies the following three conditions:

- **additive identity:** $0 \in U$
- **closed under addition:** $u, w \in U \implies u + w \in U$;
- **closed under scalar multiplication:** $a \in F$ and $u \in U \implies au \in U$.

If $u \in U$, then $-u = (-1)u$ is also in U by the third condition above. Hence every element of U has an additive inverse in U.

Example of subspaces

- If $b \in F$, then
$$\{(x_1, x_2, x_3, x_4) \in F^4 : x_3 = 5x_4 + b\}$$
is a subspace of F^4 if and only if $b = 0$.
- The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $R^{[0,1]}$.
- The set of differentiable real-valued functions on R is a subspace of R^R.
- The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $R^{(0,3)}$ if and only if $b = 0$.
- The set of all sequences of complex numbers with limit 0 is a subspace of C^∞.

The subspaces of R^2 are precisely $\{0\}, R^2$, and all lines in R^2 through the origin. The subspaces of R^3 are $\{0\}, R^3$, and all the lines in R^3 passing through the origin, and all planes in R^3 through the origin.

A couple things before we finish talking about primitive subspaces: Clearly $\{0\}$ is the smallest subspace of V and V itself is the largest subspace of V. The empty set is not a subspace of V because a subspace must be a vector space and hence must contain at least one element, namely, an additive identity.

3.0.2 Sum of subspaces

The union of subspaces is rarely a subspace, which is why we usually work with sums rather than unions.

Sum of Subsets Suppose $U_1, \ldots, U_m$ are subsets of V. The ***sum*** of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sum of elements of $U_1, \ldots, U_m$. More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}.$$

Example Suppose U is the set of all elements of F^3 whose second and third coordinates equal 0, and W is the set of all elements of F^3 whose first and third coordinates equal 0:

$$U = \{(x, 0, 0) \in F^3 : x \in F\} \qquad \text{and} \qquad W = \{(0, y, 0) \in F^3 : y \in F\}.$$

Then

$$U + W = \{(x, y, 0) : x, y \in F\}.$$

Example Suppose that $U = \{(x, x, y, y) \in F^4 : x, y \in F\}$ and $W = \{(x, x, x, y) \in F^4 : x, y \in F\}$. Then

$$U + W = \{(x, x, y, z) \in F^4 : x, y, z \in F\}.$$

Now this one relies on the fact that we named x in $U + W$ as $x + x$, y as $x + y$ and z as $y + y$.

3.0.3 Sum of subspaces is the smallest containing subspace

Suppose $U_w, \ldots, U_m$ are the subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

Proof It is easy to see that $0 \in U_1 + \cdots + U_m$ and that $U_1 + \cdots + U_m$ is closed under addition and scalar multiplication. Thus $U_1 + \cdots + U_m$ is a subspace of V.

Clearly $U_1, \ldots, U_m$ are all contained in $U_1 + \cdots + U_m$ (to see this, consider sums $u_1 + \cdots + u_m$ wheree all except one of the u's are 0). Conversely, every subspace of V containing $U_1, \ldots, U_m$ contains $U_1 + \cdots + U_m$ (because subspaces must contain all finite sums of their elements). Thus $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

3.0.4 Direct Sums

Suppose $U_1, \ldots, U_m$ are subspaces of V. Every element of $U_1 + \cdots + U_m$ can be written in the form

$$u_1 + \cdots + u_m,$$

where each u_j is in U_j.