AoPS Good Questions

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I will include questions from **Volume 2** of the series which take me more than an hour on average to solve fully. I won't be drawing the diagrams for geometry problems because it is time consuming and only add a little.

I have also added the more useful points of each chapter that are worth revising.

I am making a note on my progress and I have realized that I have great trouble remembering the formula for the binomial expansion and sum of geometric series.

§1 Cyclic Quadrilaterals

Problem 1 (*Bulgaria 1993*)**.** A parallelogram $ABCD$ with an acute angle BAD is given. The bisector of $\angle BAD$ intersects CD at point L, and the line BC at point K. Let O be the circumcenter of ΔLCK. Prove that the quadrilateral $DBCO$ is inscribed in a circle.

Proof. Since our goal is to prove $DBCO$ as a cyclic quadrilateral, we should either

- Prove that the two angles are equal or,
- The opposite angles of $DBCO$ are supplementary.

We should go with the first since it's easier to prove that the angles are equal because of congruency and also because parallelograms and angle bisectors have a lot of symmetry so there should be plenty of reasons to convince yourself to construct another line DC and prove $\Delta DCO \cong \Delta BKO$. □

Problem 2. Prove that if in $ABCD$ we let $a = AB$, $b = BC$, $c = CD$, and $d = DA$, we have

$$[ABCD]^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{B+D}{2}\right),$$

where s is the semiperimeter of the quadrilateral.

Proof.

$$[ABCD]^2 = \frac{1}{4}(a^2b^2\sin^2 B + 2abcd\sin B\sin D + c^2d^2\sin^2 D)$$
$$= \frac{1}{4}\left(a^2b^2(1-\cos^2 B) + 2abcd\sin B\sin D + c^2d^2(1-\cos^2 D)\right)$$

and so we can solve it easily after this step.

□

§2 Polynomials

A few points to note:

- **Synthetic division** only works when you have divisor in the form of $x - a$, as in the coefficient of x is 1 and it is a linear polynomial.
- For any $f(x)$, $f(a)$ is the remainder when $f(x)$ is divided by $x - a$.
- If a is a root of $f(x)$ then $(x - a) \mid f(x)$. And so $f(a) = 0$ (the literal definition of a root).
- Every polynomial with n coefficients has n roots.
- The **Rational Root Theorem** states that any polynomial $f(x)$ where

$$f(x) = a_n x^n + a_{n-1} x^n - 1 + \cdots + a_0$$

  with integer coefficients has all rational roots of the form p/q where $|p|$ and $|q|$ are relatively prime, and

$$\mathbf{p} \mid \mathbf{a_0},\ \mathbf{q} \mid \mathbf{a_n}.$$

- If a_p and a_n represent the number of positive and negative roots of a polynomial $f(x)$ then:

$$a_p \leq \text{ total sign changes in } f(x), a_n \leq \text{ total sign changes in } f(-x).$$

  The bounds of roots c_U and c_L can be determined by the nature of coefficients (positive or negative) of $f(x)/(x - c_U)$ or $f(x)/(x - c_L)$ respectively.
  This is the **Descartes' Rule of Signs**.
- We should also note that if a polynomial has a complex number as a root, then it's conjugate is also a root. (This can be shown using properties of complex numbers such as $\overline{w + z} = \bar{w} + \bar{z}$, $\overline{z^k} = (\bar{z})^k$, and $w = z \implies \bar{w} = \bar{z}$.
- For the general polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, the **$k$th symmetric sum** of the roots is $(-1)^k a_{n-k}/a_n$. Here the **kth symmetric sum** refers to the sum of all products of the roots taken k at a time.
- Polynomial $g(x)$ with roots that are **reciprocal** to the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is

$$g(x) = x^n f(1/x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

- Polynomial $g(x)$ with roots that are k **times** the roots of polynomial $f(x)$ can be written as

$$g(x) = k^n f(x/k) = a_n x^n + k a_{n-1} x^{n-1} + \cdots + k^{n-1} a_1 x + k^n a_0$$

- Polynomial $g(x)$ with roots k **more** than the roots of polynomial $f(x)$ can be written as

$$g(x) = f(x - k)$$

- If we let s_m be the sum of the mth powers of the roots of $f(x) = a_n x^n + \cdots | a_0$, then the **Newton's sums** can be written as

$$\begin{aligned} a_n s_1 + a_{n-1} &= 0 \\ a_n s_2 + a_{n-1} s_1 + 2a_{n-2} &= 0 \\ a_n s_3 + a_{n-1} s_2 + a_{n-2} s_1 + 3a_{n-3} &= 0 \\ a_n s_4 + a_{n-1} s_3 + a_{n-2} s_2 + a_{n-3} s_3 + 4a_{n-4} &= 0 \end{aligned}$$

Problem 3. If $p(1)$ and $p(0)$ are odd, then the polynomial $p(x)$ has no integer roots.

Proof.

$$p(0) \equiv 1 \pmod 2 \implies p(2n) \equiv 1 \pmod 2,$$
$$p(1) \equiv 1 \pmod 2 \implies p(2n+1) \equiv 1 \pmod 2, n \in \mathbb{N},$$

Hence no integer value n can make $p(n) = 0$ and hence it has no integral roots. □

Problem 4. Let r,s,and t be the roots of $x^3 - 6x^2 + 5x - 7 = 0$. Find

$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2}$$

Proof. I am embarassed that it took me so long to figure out that synthetic division wouldn't work.

Firstly, we need to observe that the sum they're asking is no other but the sum of the squares of the reciprocals of the roots. The function can thus be rewritten as

$$-7x^3 + 5x^2 - 6x + 1 = 0$$

by rearranging the coefficients. Now, one can write the Newton's sums for this polynomial as

$$-7s_2 + 5s_1 + 2(-6) = 0 \implies -7s_2 + 5 \cdot \frac{5}{7} - 12 = 0$$

and hence we get $s_2 = -52/49$.

□

Problem 5 (*Canada 1975 P8*)**.** Let k be a positive integer. Find all polynomials with real coefficients which satisfy the equation

$$P(P(x)) = [P(x)]^k$$

Proof. Since P is a polynomial, it is either constant or it takes infinitely many values. For the first case, we have $c = c^k$, so either $k - 1$ and c is arbitrary or $k > 1$ and $c = -1$ (if k is odd), $c = 0$ or $c = 1$.

For non-constant $P(x)$, let $z = P(x)$. We then have $P(z) = z^k$ for infinitely many values of z.

Hence the polynomial $P(x) - z^k$ has infintely many roots but a finite degree (k), so the polynomial must be zero everywhere, or $P(z) - z^k = 0$, so that $P(x) = x^k$ is the only family of non-constant polynomials which solve the given equation. □

Problem 6 (*MOP*)**.** For $n > 1$ let a_1, a_2,..., a_n be n distinct integers. Prove that the polynomial

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_n) - 1$$

cannot be written as $g(x)h(x)$ where g and h are nonconstant polynomials with integer coefficients.

Proof. Since $g(x)h(x) = f(x)$, $\forall a_1, a_2, \ldots, a_n$, $g(a_i)h(a_i) = f(a_i) = -1$. Since both $g(a_i)$ and $h(a_i)$ are integers, they must be -1 and 1.

As $a_1, \ldots, a_n$ are all solutions to $g(x)h(x)$, a polynomial $q(x) = g(x) + h(x) = 0$ has n roots.

Now $g(x) \cdot h(x) = f(x) \implies \deg(g(x) \cdot h(x)) = \deg(f(x)) = n$ which implies that neither $g(x)$ or $h(x)$ have the degree more than $n - 1$.

Therefore, $\deg(g(x) + h(x) = q(x)) \leq n - 1$. It follows that either $q(x)$ is a zero polynomial or $\deg(q(x)) < n$. If $q(x) = 0$, then $g(x) = -h(x)$ and so the leading term of $f(x)$ gets a negative coefficient which is not true. Hence $\deg(q(x)) < n$ which implies that $q(x)$ has less than n roots. This is a contradiction to a statement we proved earlier. Hence, we cannot factor $f(x)$ as stated in the problem.

□

§3 Functions

- Function $g(x)$ is the inverse of the function $f(x)$ if $g(f(x)) = x$ and so, $f(x)$ is also the inverse of $g(x)$.
- Functions may satisfy some identities in a problem. For example, if $f(x) = \log x$ then

  $$f(xy) = f(x) + f(y)$$

  and similarly if $f(x) = \sin x$ and $g(x) = \cos x$ then

  $$f(x + y) = f(x)g(y) + f(y)g(x)$$

  and

  $$[f(x)]^2 + [g(x)]^2 = 1$$

- The real trick is when you're given an identity, and then you have to find the functions that satisfy it. This can be done using a few general techniques discussed below.

  (i) The method of **Isolation** is exemplified in solving an identity with x and y such as

  $$yf(x) = xf(y)$$

  It starts by *isolating* the x and the y terms

  $$\frac{f(x)}{x} = \frac{f(y)}{y}$$

  We can define a new function $g(t) = f(t)/t$ and so the above equation becomes

  $$g(x) = g(y)$$

  which can only be possible if $g(x)$ is a constant, and so $\frac{f(x)}{x} = c$ for some constant c. Hence, $f(x) = cx$ is the family of solutions.

  > **Remark 3.1.** Once we have the family of solutions, we need to show that *every* function of this form is a solution[a]
  >
  > For this example, we can do this by going back to the original equation $yf(x) = xf(y)$ and **substitute** in the functional form. Then $f(x)$ becomes cx and $f(y)$ becomes cy, making our relation $ycx = xcy$. Thus any function $f(x) = cx$ does

> the job.
>
> [a] This is pretty intuitive

(ii) The next method of solving functional equations is **Substitution** which works by substituting one of the input variables in order to simplify the equation.

> **Remark 3.2.** Substitution is usually used when dealing with 2 variables and can come in handy when dealing with questions that want you to find a general equation 'in the form of some function'.

(iii) The last, and the most important one to remember , is the method of using **Cyclic functions** wherein you can the function as an input of itself which can be used to manipulate these equations and be solved by using operations on the equation(s).

Here is a more general way of putting the same:

$$g(g(\cdots g(x)\cdots)) = x$$

for some number of nested g's. For example $1/x$ is cyclic with **order** 2 (the number of nested functions).

- If the functional equations are more *general* then we should consider using **arbitrary functions** which simplify the functions by generalizing them using new functions, and continue this process until we can find a condition for which a function can occur. For example, if at the end, we get a function of the form $h(x) = h(-x)$, we can call it a day since $h(x)$ can only equal to the RHS if it's an even function.

> **Remark 3.3.** Rather than just having an arbitrary constant, our solution can have an arbitrary *function*, because h can be chosen any way we like (as long as it's even).

Problem 7 (IMO Shortlist 1989)**.** Let $g : \mathbb{C} \to \mathbb{C}$, $\omega \in \mathbb{C}$, $a \in \mathbb{C}$, $\omega^3 = 1$, and $\omega \neq 1$. Show that there is one and only one function $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) + f(\omega z + a) = g(z), z \in \mathbb{C},$$

and find the function f.

Solution. Since ω is one of the **third roots of unity** and it's not 1, we can write $\omega^2 + \omega + 1 = 0$ and use the method of cyclic functions to write $z = \omega z + a$ s.t.

$$f(\omega z + a) + f(\omega^2 z + \omega a + a) = g(\omega z + a) \tag{1}$$

and use it again,

$$f(\omega^2 z + \omega a + a) + f(\omega^3 z + \omega^2 a + \omega a + a) = g(\omega^2 z + \omega a + a). \tag{2}$$

Since, $\omega^2 + \omega + 1 = 0$ and $\omega^3 = 1$, we can substitute them to write

$$f(\omega^2 z + \omega a + a) + f(z) = g(\omega^2 z + \omega a + a). \tag{3}$$

Upon adding equations the OG given equation, (1), and (3), we get

$$f(z) + f(\omega z + a) + f(\omega^2 z + \omega a + a) = [g(\omega z + a) + g(\omega^2 z + \omega a + a) + g(z)]/2, \tag{4}$$

and subtracting (1) we get,

$$f(z) = [g(z) - g(\omega z + a) + g(\omega^2 z + \omega a + a)]/2.$$

Since this equation specifies $f(z)$ specifically in terms of $g(z)$, this is the one and only one $f(z)$ which solves the given funcitonal equation. □

> **Remark 3.4.** This question was especially helpful for my growth since I wasn't aware about the third roots of unity, and it also showed me another example of using cyclic function manipulation. Thanks IMO Shortlist.

§4 Complex Numbers

Definition 4.1 (Trigonometric representation)**.** The trigonometric representation of a complex number, or simply $z = r \operatorname{cis} \theta$ where cis is $\cos\theta + i \sin\theta$ is another way of representing complex numbers with polar parameters (r,θ). (r,θ) is also called the **Polar representation** of a complex number.

- Since $z = r\cos\theta + ir\sin\theta$, the polar representation of i is $(1,\pi/2)$ since $r\cos\theta = 0$ and $r\sin\theta = 1$ s.t. θ must be $\pi/2$ and r must be 1.
- The **complex absolute value** is equal to the root of the sum of the squares of $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$. So $|x+iy| = \sqrt{x^2+y^2}$. A few properties to note:
  1. $|zw| = |z||w|$ for any $z,w \in \mathbb{C}$,
  2. similarly, $|z/w| = |z|/|w|$,
  3. but, $|z+w| \neq |z| + |w|$.
  4. Intuitively, $|w| + |z| \geq |w+z|$ is always true and is called the **Triangle Inequality** for complex numbers.
- $z_1 z_2 = r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$
- $[r(\cos\theta + i\sin\theta)]^2 = r^n(\cos(n\theta) + i\sin(n\theta))$ which can be written as $(r,\theta)^n = (r^n, n\theta)$ and so $(r,\theta)^{-n} = (r^{-n}, -n\theta)$.

Definition 4.2 (DeMoivre's Theorem)**.** For $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, $(\cos x + i\sin x)^n = \cos(nx) + i\sin(nx)$. For $x \in \mathbb{R}$ and $n \in \mathbb{Q}$, let $n = p/q$ s.t.

$$(r,\theta)^{p/q} = (r^p, p\theta)^{1/q}.$$

One qth root of the above equation is $(r^{p/q}, \frac{p}{q}\theta)$ since $\left((r^{p/q})^q, q(p\theta/q)\right) = (r^p, p\theta)$.

Since (r,θ) representation is valid for any z and any $\theta + 2\pi k$ (for $k \in \mathbb{Z}$), we can write the qth roots of the polar form in a more generalized way

$$\left(r^{p/q}, \frac{p\theta}{q} + \frac{2\pi k}{q}\right).$$

Proof. Consider some general complex number (s,ϕ) such that $(r,\theta)^{p/q} = (s,\phi)$. We can write this as $(r,\theta)^p = (s,\phi)^q$, and so $(r^p, p\theta) = (s^q, q\phi)$. This can only be possible if

$$r^p = s^q \text{ and } p\theta + 2\pi k = q\phi$$

which clearly forces (s,ϕ) to be of the desired form. □

Example 4.3
Find all cube roots of $4 + 4\sqrt{3}i$.

Solution. $4 + 4\sqrt{3}i = 8(\cos 60^\circ + i\sin 60^\circ)$ giving us the polar coordinates $(8, 60^\circ)$.
Now, by **DeMoivre's Theorem**, we can write the cube root (1/3) polar form as

$$(8^{1/3}, 20^\circ + 120^\circ k)$$

Thus we have the roots as $(2, 20^\circ)$, $(2, 140^\circ)$, and $(2, 260^\circ)$ for $k = 0$, 1, and 2 respectively. □

i

§4.1 Exponential Form

This section deals with one of the former equations we've seen in definition 4.2, but here we go again.

$$(cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

So for a function $f(\theta) = \cos\theta + i\sin\theta$ we can write the above equation as

$$[f(\theta)]^n = f(n\theta).$$

Since such a function needs to be exponential ($(a^x)^n = a^{nx}$), Euler gave the following identity

$$e^{iy} = \cos y + i\sin y.$$

This relation is very useful and we can write the **exponential representation/form** of a complex number to be $re^{i\theta}$, where only r and θ are to be known.
We can use this to prove DeMoivre's Theorem, as

$$\left(re^{i}\theta\right)^n = r^n e^{ni\theta}.$$

For a rational exponent q, we can write it as

$$\left(re^{i(\theta+2\pi k)}\right)^g = r^g e^{gi\theta+2\pi ikg}.$$

Example 4.4
Find i^i.

Solution. We can see how powerful the exponential form is with this example. We can write $i = e^{i(\pi/2+2\pi k)}$, and so $i^i = e^{-(\pi/2+2\pi k)}$.
Thus there are infinite values for i^i for every real value of k.

$$\ldots, e^{\frac{-5\pi}{2}}, e^{\frac{-\pi}{2}}, e^{\frac{3\pi}{2}}, e^{\frac{7\pi}{2}}, \ldots$$

□

Problem 4.5. Prove the DeMoivre's Theorem for a complex power.

Solution. So we continue from the proof of the DeMoivre's theorem where we have $(r, \theta)^{p/q} = (s, \phi)$ but instead of p/q we put $-1/i$ which is the same as i.

So we have

$$(r, \theta)^{-1} = (s, \phi)^i \implies (r^{-1}, -\theta) = (s^i, i\phi)$$
$$\implies r^{-1} = s^i \text{ and, } -\theta = i\phi + 2\pi k.$$

Since the LHS of both sides is true, the RHS is also true.

□

> **Remark 4.6.** I don't know if this proof is true and I cannot find any solution for it. I will ask the Discord but for now I will take this to be correct?

- $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$.
- Adding the two equations and dividing by 2, we get

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

§4.2 Hyperbolics

Using the above two equations for sine and cosine, we can define two new functions, the **hyperbolic sine** and the **hyperbolic cosine** $\sinh x$ and $\cosh x$ which are defined below

$$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}.$$

A few properties are using hyperbolic sines and cosines are mentioned below

- $\cosh^2 x - sinh^2 x = 1$.
- $\sinh 2x = 2 \sinh x \cosh x$.
- $\cosh 2x = 2(1 + \sinh^2 x)$.

§4.3 Roots of Unity

This is an important section. DeMoivre's theorem can be easily used to find the n nth roots of 1. Since $1 = e^{2\pi k i}$ for integers k, the nth roots are given by

$$1^{1/n} = e^{2\pi i k/n} = 1, \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}, \cos\frac{4\pi}{n} + i \sin\frac{4\pi}{n}, \ldots$$

> **Remark 4.7.** The nth roots can all be plotted in the complex plane in a unit circle, and as their angles lie between 0 and 2π, they form a regular polygon.

Another property of the nth roots of unity is that they satisfy the equation

$$x^n - 1 = 0.$$

SInce we know 1 is a root, $(x - 1)$ divides the polynomial on the left, which we can then factor as

$$(x - 1)(x^{n-1} + x^{n-2} + \cdots + 1) = 0$$

So for any root $\omega \neq 1$, we have

$$\omega^{n-1} + \omega^{n-2} + \cdots + 1 = 0$$ [1]

Problem 4.8 (AHSME 1981). If θ is a constant such that $0 < \theta < \pi$ and $x + \frac{1}{x} = 2\cos\theta$, then for each positive integer n, find $x^n + \frac{1}{x^n}$ in terms of n and θ.

Solution. We have $x + \frac{1}{x} = 2\cos\theta \implies x^2 - 2x\cos\theta + 1 = 0$.

Now we can find the roots of this quadratic using the quadratic formula. We get the roots to be

$$\cos\theta \pm i\sin\theta.$$

Therefore we can plug this into the exponential form of complex and get the following

$$x = e^{i\theta} \implies x^n = e^{ni\theta},$$

such that $x^{-n} = e^{-ni\theta}$.

Plugging these values into the equation we need to find

$$x^n + \frac{1}{x^n} = e^{ni\theta} + e^{-ni\theta} = \cos n\theta \pm i\sin n\theta + \cos\theta \mp i\sin n\theta = 2\cos n\theta$$

□

Problem 4.9. Evaluate

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n},$$

where $\cos\theta = 1/5$.

Solution. Note that $\cos(n\theta) = \Re(e^{in\theta})$ and so we can write the sum, which we denote as S, as

$$S = \Re\left(\sum_{n=0}^{\infty} \frac{e^{ni\theta}}{2^n}\right) = \Re\left(\sum_{n=0}^{\infty} \left(\frac{e^{i\theta}}{2}\right)^n\right).$$

Since n tends to infinty, the sum S can be written as the geometric sum with 1 as the initial term and $e^{i\theta}/2$ as the common ratio.

Hence the sum is equal to

$$S = \Re\left(\frac{1}{1 - e^{i\theta}/2}\right) = \Re\left(\frac{1}{1 - (\cos\theta)/2 - (i\sin\theta)/2}\right).$$

To determine the real part of this complex number, we must rationalize it, yielding

$$S = \Re\left(\frac{1 - \frac{1}{2}\cos\theta + \frac{i}{2}\sin\theta}{(1 - \frac{1}{2}\cos\theta)^2 + \frac{1}{4}\sin^2\theta}\right) = \frac{1 - \frac{1}{2}\cos\theta}{(1 - \frac{1}{2}\cos\theta)^2 + \frac{1}{4}\sin^2\theta}.$$

Plugging in the value of $\cos\theta = 1/5$ and $\sin\theta = 24/25$, we get

$$S = \frac{1 - 1/10}{(9/10)^2 + (1/4)(24/25)} = \frac{6}{7}.$$

□

Problem 4.10 (China TST and MOP). Suppose that the coefficients of the equation $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$ are real and satisfy $0 < a_0 \le a_1 \le \cdots \le a_{n-1} \le 1$. Let z be a complex root of the equation with $|z| \ge 1$. Show that $z^{n+1} = 1$.

Solution. Since z is a root of the equation, let $x = z$ and so

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

To get z^{n+1}, we'll multiply this equation with $z - 1$

$$(z-1)(z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0) = 0$$
$$\implies z^{n+1} + z^n(a_{n-1} - 1) + \cdots + z(a_0 - a_1) - a_0 = 0.$$

Solving for z^{n+1} we get,

$$z^{n+1} = z^n(1 - an - 1) + \cdots + z(a_1 - a_0) + a_0$$

Now we can apply the triangle inequality by which

$$|z|^{n+1} \le |z^n(1 - a_{n-1})| + \cdots + |z(a_1 - a_0)| + |a_0|$$
$$\implies |z|^{n+1} \le |z|^n(1 - a_{n-1}) + \cdots + |z|(a_1 - a_0) + a_0.$$

Since $|z| \ge 1$, $|z|^i \ge |z|^j$, for all $i \ge j$, the degree of the right side is upto n.
Hence we can write the inequality as

$$|z|^{n+1} \le |z|^n(1 - a_{n-1}) + \cdots + |z|^n(a_1 - a_0) + |z|^n a_0.$$

Therefore, $|z|^{n+1} \le |z|^n$. This is only possible if $|z| = 1$. Now, our triangle inequality becomes an equality like

$$|z|^{n+1} = |z|^n(1 - a_{n-1}) + \cdots + |z|(a_1 - a_0) + a_0.$$

But, a triangle inequality can only become an equality **if** all numbers on one side have the same angles. Since, a_0 is real, all other terms on the RHS must also be real.

Thus, the LHS becomes real, and hence, $|z| = z = 1 \implies z^{n+1} = 1$. □

§5 Vectors and Matrices

- The length of the vector is denoted by $\|\vec{v}\|$.
- The length of the vector $\vec{w} - \vec{v}$ can be found using the law of cosines. Since $\vec{v}$, $\vec{w}$, and $\vec{w} - \vec{v}$ form a triangle whose sides have lengths $\|\vec{v}\|$, $\|\vec{w}\|$, and $\|\vec{w} - \vec{v}\|$, we have

  $$\|\vec{w} - \vec{v}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos\theta,$$

  where θ is the angle between $\vec{v}$ and $\vec{w}$.
- The expression $\|\vec{v}\|\|\vec{w}\| \cos\theta$ in the above point is called the **dot product** of $\vec{v}$ and $\vec{w}$; it is denoted as $\vec{v} \cdot \vec{w}$.
- The formula for the dot product then becomes

  $$\vec{v} \cdot \vec{w} = \frac{\|\vec{v}\|^2 + \|\vec{w}\|^2 - \|\vec{w} - \vec{v}\|^2}{2}.$$
- Some properties of the dot product:
  1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ (**Commutative**)
  2. $\vec{v} \cdot \vec{w} = 0$ for nonzero $\vec{v}$ and $\vec{w}$ if and only if $\vec{v}$ and $\vec{w}$ are perpendicular.

3. $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$ for any real number c.
4. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$. (**Distributive**)

- We associate a vector with coordinates of its head. If the head coordinates are (x, y), the vector is represented as $\begin{pmatrix} x & y \end{pmatrix}$ or as $\begin{pmatrix} x \\ y \end{pmatrix}$. The former is called a **row vector** and the latter a **column vector**.

  For example, the vector sum $\begin{pmatrix} x_1 & y_1 \end{pmatrix} + \begin{pmatrix} x_2 & y_2 \end{pmatrix}$ is just

  $$\begin{pmatrix} (x_1 + x_2) & (y_1 + y_2) \end{pmatrix}.$$

- The dot product is written rather nicely in the coordinate form. For example, let $\vec{v_1} = \begin{pmatrix} x_1 & y_1 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} x_2 & y_2 \end{pmatrix}$ which form angles θ_1 and θ_2 with the positive x axis. Their dot product is then

  $$\begin{aligned} \vec{v_1} \cdot \vec{v_2} &= \|\vec{v_1}\|\|\vec{v_2}\| \cos(\theta_1 - \theta_2) \\ &= \|\vec{v_1}\|\|\vec{v_2}\|(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) \\ &= (\|\vec{v_1}\| \cos\theta_1)(\|\vec{v_2}\| \cos\theta_2) + (\|\vec{v_1}\| \sin\theta_1)(\|\vec{v_2}\| \sin\theta_2) \\ &= x_1 x_2 + y_1 y_2. \end{aligned}$$

i

§6 Analytic Geometry

Very annoying chapter but a lot of theory to learn.

- The tangent of the angle between any two lines (β) is

  $$\tan\beta = \tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta \tan\alpha} = \frac{m_2 - m_1}{1 + m_1 m_2}$$

  where α is the angle between the x axis and one of the lines, and θ is the angle between the second line and the x axis and so $\theta = \alpha + \beta$.

- The vector which is **normal** to any line $Ax + By + C = 0$ is $\begin{pmatrix} A & B \end{pmatrix}$.

- The distance from a point (x_0, y_0) to the line $Ax + By + C = 0$ can be written as

  $$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

Definition 6.1 (Shoelace Algorithm)**.** If we have a **convex polygon** (meaning any segment connecting two points in the polygon lies entirely within the polygon) with n vertices in order at $(x_1, y_1), \dots, (x_n, y_n)$, then we can find the area as follows.

1. List the vertices vertically in order, putting the first point both first and last as shown

$$\begin{matrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{matrix}$$

2. Then find the products along the diagonals as shown

$$\begin{matrix} x_2y_1 & x_1y_2 \\ \vdots & \vdots \\ x_{n-1}y_{n-2} & x_{n-2}y_{n-1} \\ x_ny_{n-1} & x_{n-1}y_n \\ x_1y_n & x_ny_1 \end{matrix}$$

3. Let S_l be the sum of the left column of products and S_r be the sum of the right column. The area then is $|S_r - S_l|/2$.

Proof. This can be proved by using this and calculating the area of a triangle, and then use induction. □

Theorem 6.2 (Pick's theorem)

The area contained in the polygon is

$$I + \frac{B}{2} - 1,$$

where I are the number of lattice points within the polygon (Interior Points) and B be the number of points which lie on the boundary of the polygon (Boundary points.)

Remark 6.3. We can prove the pick's theorem for any n-gon by dividing it into $n-2$ triangles. We can also prove the pick's theorem for any non-convex polygons by chopping off a convex polygon from another.

- The distance formula for n dimensional points is similar to the distance formula between two points on the cartesian plane.

  The distance between two points A and B in an n dimensional graph with dimensions a_1,a_2,..., a_n is

$$AB = \sqrt{(a_1^A - a_1^B)^2 + (a_2^A - a_2^B)^2 + \ldots + (a_n^A - a_n^B)^2}.$$

  Why? Because any distance can be derived from the pythagoras formula using the $n-1$ dimensional distances iteratively.

Example 6.4

Suppose we have a line through $(1, 2, 3)$ in the direction of $\vec{v}$, where $\vec{v} = \begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$. Hence, $(1, 2, 3) + 2(2, 3, 4) = (5, 8, 11)$. is on the line.

In fact, we get a point on the line by adding any multiple $t\vec{v}$ of $\vec{v}$ to $(1, 2, 3)$. Our line then becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = t \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

Hence we have described the line as a set of parametric equations:

$$x = 1 + 2t$$
$$y = 2 + 3t$$
$$z = 3 + 4t$$

If we solve each of these for t, we can write

$$t = \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

In this notation, we don't need to include the t to describe the line.

- Just like our example of calculating the distance between a point (x_0, y_0) and a line $Ax + By + C = 0$, we can find the distance between a point (x_0, y_0, z_0) and a plane $Ax + By + Cz + D = 0$ using the formula

$$Distance = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

§6.1 Volumes

- The product $\vec{c} \cdot (\vec{a} \times \vec{b})$ is often called the **box product** of the three vectors, and the magnitude is the same regardless of the order of $\vec{a}$, $\vec{b}$, and $\vec{c}$ in the product, and it represents the volume of the parallelpiped with sides OA,OB, and OC.

> **Remark 6.5.** Here the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors taken from O to A, O to B, and O to C respectively.

Example 6.6

Find the volume of the tetrahedron with vertices $(1, 0, -2)$, $(2, 3, 1)$, $(2, 1, -4)$, and $(-1, 2, -1)$.

Solution. We begin by forming the equation to calculate the volume of a tetrahedron.

Claim 6.7 — The volume of the tetrahedron with vertices O,A,B, and C with the base ΔAOB is

$$[\vec{c} \cdot (\vec{a} \times \vec{b})]/6.$$

Proof. We already know that the volume of a parallelpiped with sides OA,OB, and OC is $\vec{c} \cdot (\vec{a} \times \vec{b})$.

Since the area of a triangle is half the area of the parallelogram with sides OA and OB and a pyramid has volume $(1/3)\times$(base area)$\times$(height) [2], the volume of the tetrahedron is then the formula in our claim. □

Letting the first point be O, our vectors are $\vec{a} = \begin{pmatrix} 1 & 3 & 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 & 1 & -2 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} -2 & 2 & 1 \end{pmatrix}$.

Now, instead of calculating $(\vec{a} \times \vec{b})$ as

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 3 \\ 1 & 1 & -2 \end{vmatrix},$$

we replace the $\vec{i}$, $\vec{j}$, and $\vec{k}$ with the components of $\vec{c}$. This is because in the dot product $\vec{c} \cdot (\vec{a} \times \vec{b})$, we are merely multiplying the components of $\vec{c}$ one by one with the components of $\vec{a} \times \vec{b}$.

Therefore, using the equation in the claim 6.1, we calculate

$$\frac{\vec{c} \cdot (\vec{a} \times \vec{b})}{6} = \begin{vmatrix} -2 & -2 & 1 \\ 1 & 3 & 3 \\ 1 & 1 & -2 \end{vmatrix} /6 = \frac{13}{3}.$$

□

- In the above example, we showed that the box product of any three vectors, $\begin{pmatrix} x_1 & y_1 & z_1 \end{pmatrix}$, $\begin{pmatrix} x_2 & y_2 & z_2 \end{pmatrix}$, $\begin{pmatrix} x_3 & y_3 & z_3 \end{pmatrix}$, is equal to

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

- A more generalized formula for the volume of a tetrahedron with vertices (x_1, y_1, z_1), (x_2, y_2, z_2),(x_3, y_3, z_3), and (x_4, y_4, z_4) is

$$\begin{vmatrix} (x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \\ (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \end{vmatrix} /6.$$

[2] Here, we are taking the base area to be the area of the parallelogram and the height to be $\|\vec{c}\|$

> **Remark 6.8.** The rows of the matrix can be written any way. This means that the second row could be the first, first the second and so on, since the determinant doesn't change.

§6.2 Curved Surfaces

Since, the equation of a circle is a function of x and y, for every value of z, we have a circle of radius r, resulting in an infinite cylinder whose axis is the z axis and radius has length r.

So we can write the parameters z and θ as

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z. \end{aligned}$$

Therefore, just as polar coordinates (r, θ) can be used to define any point in the plane, the parameters (r, θ, z) can be used to identify any point in space. We can call these the **cylindrical coordinates**.

§6.2.1 Spheres

From the distance formula, for any point (x, y, z) on the sphere we have

$$\rho = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}.$$

Squaring this we have the general form for a sphere:

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \rho^2,$$

where (x_0, y_0, z_0) defines the center of the sphere.

- For a sphere $x^2 + y^2 + z^2 = \rho^2$, we can write the **spherical coordinates** as

$$\begin{aligned} x &= \rho\cos\theta\sin\phi \\ y &= \rho\sin\theta\sin\phi \\ z &= \rho\cos\phi. \end{aligned}$$

  where ϕ is the angle between the z axis and ρ (the sphere radius), θ is the angle between the projection of ρ onto the xy plane and the x axis.

§6.3 Vectors and Geometry

- The expression for the centroid of ΔABC is

$$\vec{G} = \frac{\vec{A} + \vec{B} + \vec{C}}{3}.$$

- The expression of the orthocenter of ΔABC is

$$\vec{H} = \vec{A} + \vec{B} + \vec{C}.$$

> **Remark 6.9.** From the proof of this expression, we learnt that it is best to take the origin for the vertices of a triangle as the circumcenter (Why? Because all vertices are of same distance from it.)

Example 6.10 (IMO 1989)

Vertex A of the acute triangle ABC is equidistant from the circumcenter O and the orthocenter H. Determine all possible values for the measure of angle A.

Solution. We have $\|\vec{H}-\vec{A}\| = \|\vec{A}\| = \|\vec{A}-\vec{O}\|$. Therefore,

$$\begin{aligned}\|\vec{H}-\vec{A}\|^2 = \|\vec{A}\|^2 = \|\vec{B}+\vec{C}\|^2 = R^2 \\ \implies R^2 &= \|\vec{B}\|^2 + \|\vec{C}\|^2 + 2\cdot\|\vec{B}\|\cdot\|\vec{C}\| \\ &= \|\vec{B}\|^2 + \|\vec{C}\|^2 - \|\vec{B}-\vec{C}\|^2 + \|\vec{B}\|^2 + \|\vec{C}\|^2 \\ \implies R^2 &= CB^2 - 4R^2 \\ \implies a/R &= \sqrt{3}\end{aligned}$$

Since by the law of sines, $\frac{a}{\sin A} = 2R$, and so, $\frac{a}{\sin A} = 2R \implies \frac{\sqrt{3}R}{\sin A} = 2R \implies \sin A = \frac{\sqrt{3}}{2} \implies \angle A = 60°$. □

§7 Equations

- The **Gaussian approach** can be used to solve linear equations with 2 or more variables in a more elegant way, by writing the coefficients as elements of a matrix.

  We know that matrix rows can be manipulated in various ways so as to get answers, and this approach is termed as **Gaussian elimination**.

  > **Remark 7.1.** The need for gaussian elimination approach gets much more drastic as the complexity of the equation increases.

  Another nuance in the gaussian approach is that we can take $\underline{A}\vec{x} = \vec{b}$ where $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. And so if we multiply both sides by $\underline{A}^{-1}$, we we get $\underline{A}^{-1}\underline{A}\vec{x} = \underline{A}^{-1}\vec{b}$, and thus $\vec{x} = \underline{A}^{-1}\vec{b}$.

  Hence, we can check if a matrix (or naturally, a set of equations) has a **unique** solution by checking if the matrix is **invertible**[3].

- We know that,

$$\begin{aligned}a^2+b^2 &= (a+b)^2 - 2ab \\ a^2-b^2 &= (a-b)(a+b) \\ a^3+b^3 &= (a+b)(a^2-ab+b^2) \\ a^3-b^3 &= (a-b)(a^2+ab+b^2)\end{aligned}$$

[3]Note that a matrix $\underline{A}$ is **invertible** iff $|A| \neq 0$.

Definition 7.2. For any a, b,

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}).$$

Corollary 7.3

For any a, b,

$$a^{2n+1} - b^{2n+1} = (a+b)(a^{2n} - a^{2n-1}b + a^{2n-2}b^2 - \cdots - ab^{2n-1} + b^{2n}).$$

Obviously, $a = b$ and $a = -b$ are one of the solutions to 7.2 and 7.3 respectively.

Example 7.4

Prove that

$$\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-a)(b-c)} + \frac{1}{c(c-a)(c-b)} = \frac{1}{abc}$$

for all sets of distinct nonzero numbers $\{a, b, c\}$.

Solution. To prove the given equation, let's modify up a bit s.t.

$$\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-a)(b-c)} + \frac{1}{c(c-a)(c-b)} - \frac{1}{abc} = 0.$$

Now, we can take the LCM $abc(a-b)(a-c)(b-c)$ and use the numerator as a function of c s.t.

$$f(c) := bc(b-c) - ac(a-c) + ab(a-b) - (a-b)(a-c)(b-c).$$

Observing the function, we find out that $f(a) = f(b) = f(0) = 0$, which implies that the whole function is zero everywhere making the LHS zero. □

§7.1 Using graphing[4]

For a few problems such as:

Example 7.5

For how many positive numbers x does $\cos x = x/8$?

Solution. We can plot two graphs $y = \cos x$ and $y = x/8$ and see where they intersect. □

Exercise 7.6 (MAO 1991)**.** Find the prime factorization of $2^{22} + 1$.

Solution. Since there are no cubes, we are forced to make use of squares:

$$2^{22} + 1 = 2^{22} + 2(2^{11}) + 1 - 2(2^{11}) = (2^{11} + 1)^2 - 2^{12},$$

which we can write as

$$(2^{11} + 1)^2 - (2^6)^2 = (2^{11} + 1 + 2^6)(2^{11} + 1 - 2^6) = (2113)(1985) = (2113)(5)(397).$$

And so we get 2113, 5, and 397 as the prime factors of $2^{22} + 1$. □

[4] I've added this section since it can help us get the answer pretty easily. In 7.5, the answer is 3 and it didn't even require a minute on our behalf.

§8 Inequalities

- One of the most recommended methods of attacking inequality-based problems is using the **Trivial Inequality** which states that for any real number x, $x^2 \geq 0$.

> **Remark 8.1.** If your proof requires you to work backwards from the given equation/inequality, always show that, even if the actual proof involves going forward.

Definition 8.2 (AM-GM Inequality)**.** The arithmetic mean of a set of positive numbers a_1,a_2,a_3,. . . ,a_n, is greater than or equal to the geometric mean of those numbers.

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{2} \geq \sqrt[n]{a_1 a_2 a_3 \cdots a_n},$$

for $a_i > 0$, $i \in \{1, \ldots, n\}$.

Proof. To prove AM-GM inequality, we'll start with a **lemma**.

Lemma 8.3

For positive real numbers x and y such that $x > y$, then $x - \epsilon \geq y + \epsilon$, where ϵ is a positive real number, implies that $(x - \epsilon)(y + \epsilon) \geq xy$.

Hence, increasing and decreasing two numbers by a constant doesn't affect the average, but does affect their product.

Proof. We find that

$$(x + \epsilon)(y - \epsilon) - xy = \epsilon(x - y) - \epsilon^2 \tag{5}$$

We also see that since $x + \epsilon \geq y - \epsilon$, $x - y \geq 2\epsilon$.

Hence, going back to 5 we can write

$$(x + \epsilon)(y - \epsilon) - xy \geq 2\epsilon^2 - epsilon^2 \geq \epsilon^2 \geq 0,$$

thereby proving the lemma. □

Suppose a_1,a_2,. . . ,a_n are distinct positive real numbers (since same a_i obviously supports AM-GM) with average A and product P.

WLOG, let $a_j < A$ be the closest number in the set of numbers and $a_k > A$ be the highest number in the set of numbers. Clearly, $a_k - A \geq A - a_j \implies a_k - A - a_j \geq A \implies a_k(A - a_j) \geq a_j + (A - a_j)$.

Using 8.3, we have the average of the two numbers the same while increasing their product. Upon repeating this process with other numbers, we can make all the elements in the set equal to A, making the maximum product of all elements equal to A. Thus, the maximum possible value of the geometric mean of the set is A. [5]

> **Remark 8.4.** The maximum only occurs when all elements are equal to A (since if one or more are not equal to A, the product of the numbers can be increased by the process above).

□

[5]One key aspect of this proof was that we kept one of the sides constant while manipulating the other, using 8.3 here. This allowed us to work toward the desired inequality and easily set an upper bound for it.

Recall from Chapter 5 that the dot product of two vectors $\vec{x}$ and $\vec{y}$ is

$$\vec{x} \cdot \vec{y} = \|x\|\|y\| \cos\theta,$$

where θ is the angle between $\vec{x}$ and $\vec{y}$. Since $\cos\theta \leq 1$, $\|x\|\|y\|\cos\theta \leq \|x\|\|y\| \implies \vec{x} \cdot \vec{y} \leq \|x\|\|y\|$.

Writing $\vec{x}$ and $\vec{y}$ in terms of their Cartesian coordinates, they are $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$ and $\begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}$.

Definition 8.5 (Cauchy's Inequality)**.** Using these coordinates and squaring both sides of the inequality $\vec{x} \cdot \vec{y} \leq \|x\|\|y\|$, we have

$$(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2 \leq (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2).$$

Using the proof, we find that the condition for equality is when $\vec{x}$ and $\vec{y}$ are in the same direction, or the ration of their components is constant s.t.

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}.$$

- A quadratic polynomial $ax^2 + bx + c \geq 0$ has 1 or 0 real roots. This is because for the quadratic to have 2 roots, it needs to touch the x axis two times therefore becoming negative. But since the quadratic stays above 0 at all times, it must have ≤ 1 *real* roots.

- For **optimization** (minimization or maximization) problems, we can often apply AM-GM, Cauchy's Inequality, or the Trivial Inequality.

Example 8.6 (A simple optimization problem)

If Farmer Bob has 96 square inches of wrapping paper. What is the volume of the largest rectangular box he can wrap with the paper?

Solution. Let x,y,z be the dimensions of the rectangular box. The surface area of the box to be covered by the paper is 96 so

$$2(xy + yz + az) = 96.$$

Using AM-GM inequality, we can write

$$\frac{xy + yz + az}{3} \geq \sqrt[3]{(xy)(yz)(az)}.$$

Hence we get $xyz \leq 64$. Using the equality condition for AM-GM, we get $x = y = z = 4$ inches. This means that the box would be a cube. □

Problem 8.7. Prove the Triangle Inequality which states that

$$a + b > c$$

for the three sides of the triangle a,b, and c.

Solution. □